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SOME PROPERTIES OF SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract: The object of the present paper is to define a new subclass $\mathcal{T}^{\zeta}_{m,\lambda}(A, B, \gamma)$ of analytic functions whose non-negative coefficients from the second onwards are negative by using the differential operator $D^{\zeta}_{m,\lambda}$. We derive some interesting properties like coefficient inequalities, distortion bounds, convolution conditions and a result which unifies radii of close-to-convexity, starlikeness and convexity.

Keywords and Phrases: Analytic functions, Modified Hadamard product, Coefficient inequalities, Convolution conditions, Al-Oboudi operator.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic univalent functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U}$.

Let $\mathcal{T}$ denote the subclass of analytic functions in $\mathcal{U}$, consisting of functions whose non-zero coefficients from the second onwards are negative, that is an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.2}$$

For any two functions f and $h \in \mathcal{T}$, where f is of the form (1.2) and $h(z) = z - \sum_{k=2}^{\infty} c_k z^k$. The modified Hadamard product of f and h. is defined by $(f \bullet h)(z) = z - \sum_{k=2}^{\infty} a_k c_k z^k$.

Let $P_1(A, B)$ [3] denote the class of analytic functions in $\mathcal{U}$ which are of the form

$$\frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad -1 \leq A < B \leq 1,$$

where ω is analytic in $\mathcal{U}$, $\quad \omega(0) = 0, \quad |\omega(z)| < 1$.

For a function f in $\mathcal{T}$, the new differential operator $D^{\zeta}_{m,\lambda}$ [2] is defined by,

$$D^{\zeta}_{m,\lambda} f(z) = z - \sum_{k=2}^{\infty} \left[1 + (k-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right]^{\zeta} a_k z^k, \tag{1.3}$$

where $0 \leq \lambda \in \mathbb{R}, \quad m \in \mathbb{N} \quad and \quad \zeta \in \mathbb{N}_0$.

or equivalently we have

$$D^{\zeta}_{m,\lambda} f(z) = z - \sum_{k=2}^{\infty} \left[1 + (k-1) c^m_j(\lambda) \right]^{\zeta} a_k z^k,$$

where $c^m_j(\lambda) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$, and $0 \leq \lambda \in \mathbb{R}, \quad m \in \mathbb{N} \quad and \quad \zeta \in \mathbb{N}_0$.

Note that, for $m = 1$, we get the Al-Oboudi operator [1] and for $m = \lambda = 1$, we get the Sălăgean operator [5].

Now using the differential operator $D^{\zeta}_{m,\lambda}$ we define a new subclass $\mathcal{T}^{\zeta}_{m,\lambda}(A, B, \gamma)$ as follows.

Definition 1.1. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}^{\zeta}_{m,\lambda}(A, B, \gamma)$ if and only if*

$$\left[\frac{(1-\gamma) z (D^{\zeta}_{m,\lambda} f(z))' + \gamma z (D^{\zeta+1}_{m,\lambda} f(z))'}{(1-\gamma)(D^{\zeta}_{m,\lambda} f(z)) + \gamma (D^{\zeta+1}_{m,\lambda} f(z))} \right] \in \mathcal{P}_1(A, B), \quad z \in \mathcal{U}. \tag{1.4}$$

where $0 \leq \gamma \leq 1, \quad 0 \leq \lambda \in \mathbb{R}, \quad m \in \mathbb{N} \quad and \quad \zeta \in \mathbb{N}_0, \quad -1 \leq A < B \leq 1$.

Observe that by specializing the parameters we get the following subclasses

- $\mathcal{S}^*_1(A, B) and \quad \mathcal{K}_1(A, B)$ studied by Ganesan [3]

- $\mathcal{T}(m,\alpha)$, studied by Hur and Oh [4]
- $\mathcal{T}_{\lambda}^{m}(A,B,\gamma)$ studied by Latha and shilpa [6].

It is our goal in the following sections to prove coefficient inequalities, distortion bounds, radii of close-to-convexity, starlikeness, convexity and convolution conditions for the functions belonging to the above class.

2. Main Results

Theorem 2.1. *Let the function $f(z)$ be defined by (1.2). Then $f(z)$ is in the class $\mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]a_k \leq B-A \tag{2.1}$$

where

$$c_k(\zeta,\lambda)=[1+c_j^m(\lambda)(k-1)]^{\zeta}. \tag{2.2}$$

The result is sharp and the extremal functions are

$$f_k(z)=z-\frac{(B-A)}{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]}z^k, \quad k\geq 2. \tag{2.3}$$

Proof. Assume that the inequality (2.1) holds and let $|z|=1$. Then we have

$$\left|\frac{(1-\gamma)z(D_{m,\lambda}^{\zeta}f(z))'+\gamma z(D_{m,\lambda}^{\zeta+1}f(z))'}{(1-\gamma)(D_{m,\lambda}^{\zeta}f(z))+\gamma(D_{m,\lambda}^{\zeta+1}f(z))}-1\right| \leq$$

$$\left[1+\frac{\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]ka_k-1}{1-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]a_k}\right] \leq \frac{B-A}{1+B}.$$

Appealing, to the maximum modulus theorem, we have $f(z)\in\mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$. Conversely, assume that $f(z)\in\mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$. Then

$$\Re\left\{\frac{z-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]ka_kz^k}{z-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]a_kz^k}\right\} > \frac{1+A}{1+B}.$$

Choose values of z on the real axis so that

$$\frac{(1-\gamma)z(D_{m,\lambda}^{\zeta}f(z))'+\gamma z(D_{m,\lambda}^{\zeta+1}f(z))'}{(1-\gamma)(D_{m,\lambda}^{\zeta}f(z))+\gamma(D_{m,\lambda}^{\zeta+1}f(z))},$$

is real. Letting $z \to 1^-$ through real values, we obtain,

$$\Re\left\{\frac{1-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]ka_k}{1-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]a_k}\right\} \geq \frac{1+A}{1+B}$$

or equivalently we get,

$$1-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]ka_k$$

$$\geq \frac{1+A}{1+B}\left[1-\sum_{k=2}^{\infty}[1+c_j^m(\lambda)(k-1)]^{\zeta}[1+\gamma c_j^m(\lambda)(k-1)]a_k\right]$$

which yields (2.1).

Theorem 2.2. *Let the function f defined by (1.2) be in the class $\mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$ then*

$$\sum_{k=2}^{\infty} a_k \leq \frac{B-A}{c_2(\zeta,\lambda)(1+\gamma c_j^m(\lambda))(2B-A+1)}. \tag{2.4}$$

and

$$\sum_{k=2}^{\infty} ka_k \leq \frac{2(B-A)}{c_2(\zeta,\lambda)(1+\gamma c_j^m(\lambda))(2B-A+1)}. \tag{2.5}$$

The equality in (2.4) and (25) is attained for the function f given by (2.3).

Proof. From Theorem 2.1, and from (2.1) it follows that

$$\begin{aligned}&(1+\gamma c_j^m(\lambda))(2B-A+1)c_2(\zeta,\lambda)\sum_{k=2}^{\infty} a_k\\ &\quad\leq \sum_{k=2}^{\infty} c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]a_k\\ &\quad\leq (B-A),\end{aligned}$$

which yields (2.4).

From (1.4), we have

$$(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)\sum_{k=2}^{\infty}[k(1+B)-(1+A)]a_k \leq (B-A),$$

that is,

$$(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)\sum_{k=2}^{\infty} ka_k(1+B) \le (B-A)+(1+A)(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)\sum_{k=2}^{\infty} a_k.$$

From (2.4) the above equation becomes

$$(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)\sum_{k=2}^{\infty} ka_k(1+B) \le$$

$$(B-A)+(1+A)(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)\frac{B-A}{(1+\gamma c_j^m(\lambda))c_2(\zeta,\lambda)(2B-A+1)}.$$

thus we have the proof.

Theorem 2.3. *Let the function f defined by (1.2) be in the class $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$. Then we have*

$$|D^i_{m,\lambda}f(z)| \ge |z| - \frac{(B-A)}{c_k(\zeta-i,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|^2. \tag{2.6}$$

and

$$|D^i_{m,\lambda}f(z)| \le |z| + \frac{(B-A)}{c_k(\zeta-i,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|^2. \tag{2.7}$$

for $z \in \mathcal{U}$, where $0 \le i \le \zeta$ and $c_k(\zeta-i,\lambda)$ is given by (2.2).
The equalities in (2.6) and (2.7) are attained for the function f given by

$$f_2(z) = z - \frac{(B-A)}{(1+c_j^m(\lambda))^{\zeta}(2B-A+1)(1+\gamma c_j^m(\lambda))}z^2. \tag{2.8}$$

Proof. Note that $f \in \mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$ if and only if $D^i_{m,\lambda}f(z) \in \mathcal{T}^{\zeta-i}_{m,\lambda}(A,B,\gamma)$ and

$$D^i_{m,\lambda}f(z) = z - \sum_{k=2}^{\infty} c_k(i,\lambda)a_k z^k. \tag{2.9}$$

From Theorem 2.1, it follows that

$$c_k(\zeta-i,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))\sum_{k=2}^{\infty} c_k(i,\lambda)a_k \le$$

$$\sum_{k=2}^{\infty} c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]a_k \le B-A$$

that is,

$$\sum_{k=2}^{\infty} c_k(i,\lambda)a_k \le \frac{(B-A)}{c_k(\zeta-i,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}. \tag{2.10}$$

finally we get (2.6) and (2.7). Note that the equalities (2.6) and (2.7) are attained for the function f defined by

$$D^i_{m,\lambda}f(z) = z - \frac{(B-A)}{c_k(\zeta-i,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}z^2. \tag{2.11}$$

this completes the proof.

Corollary 2.4. *Let the function f defined by (1.2) be in the class $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$. Then we have,*

$$|f(z)| \ge |z| - \frac{(B-A)}{c_2(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|^2. \tag{2.12}$$

and

$$|f(z)| \le |z| + \frac{(B-A)}{c_2(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|^2. \tag{2.13}$$

for $z \in \mathcal{U}$. The equalities in (2.12) and (2.13) are attained for the function f_2 given by (2.8).

Corollary 2.5. *Let the function f defined by (1.2) be in the class $\mathcal{T}^{m}_{\lambda}(A,B,\gamma)$. Then we have,*

$$|f'(z)| \ge 1 - \frac{2(B-A)}{c_2(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|. \tag{2.14}$$

and

$$|f'(z)| \le 1 + \frac{2(B-A)}{c_2(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}|z|. \tag{2.15}$$

for $z \in \mathcal{U}$. The equalities in (2.14) and (2.15) are attained for the function f_2 given in (2.8).

Corollary 2.6. *Let the function f defined by (1.2) be in the class $\mathcal{T}^{m}_{\lambda}(A,B,\gamma)$. Then the unit disc is mapped onto a domain that contains the disc*

$$|\omega| < \frac{c_2(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda)) - (B-A)}{c_k(\zeta,\lambda)(2B-A+1)(1+\gamma c_j^m(\lambda))}. \tag{2.16}$$

The result is sharp with the extremal function f_2 given in (2.8).

Theorem 2.7. *Let the function* $f \in \mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$. *Then*

$$\left|\frac{f \bullet \Phi}{f \bullet \Psi} - 1\right| < 1-\delta, \quad in \quad |z| < r,$$

with $\Phi(z) = z - \sum_{k=2}^{\infty} \lambda_k z^k$, *and* $\Psi(z) = z - \sum_{k=2}^{\infty} \mu_k z^k$, *are analytic in* $\mathcal{U}$ *with the conditions* $\lambda_k \geq 0$, $\mu_k \geq 0$, $\lambda_k \geq \mu_k$, *for* $k \geq 2$ *and* $f(z) \bullet \Psi(z) \neq 0$. *Where*

$$r = \inf_k \left[\frac{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)](1-\delta)}{(B-A)[(\lambda_k-\mu_k)+\mu_k(1-\delta)]}\right]^{\frac{1}{k-1}}, \quad k \geq 2. \tag{2.17}$$

Proof. Consider,

$$\left|\frac{f \bullet \Phi}{f \bullet \Psi} - 1\right| = \left|\frac{z - \sum_{k=2}^{\infty} \lambda_k a_k z^k}{z - \sum_{k=2}^{\infty} \mu_k a_k z^k} - 1\right| \leq$$

$$\left|\frac{z - \sum_{k=2}^{\infty} \lambda_k a_k z^k - z + \sum_{k=2}^{\infty} \mu_k a_k z^k}{z - \sum_{k=2}^{\infty} \mu_k a_k z^k}\right| \leq \frac{\sum_{k=2}^{\infty} a_k[\lambda_k - \mu_k]|z|^{k-1}}{1 - \sum_{k=2}^{\infty} \mu_k a_k |z|^{k-1}} < 1-\delta$$

$$\sum_{k=2}^{\infty} a_k[(\lambda_k - \mu_k) + (1-\delta)\mu_k]|z|^{k-1} \leq 1-\delta \tag{2.18}$$

where r is given by (2.17).
Hence by using Theorem (2.1), (2.18) will be true if,

$$\frac{[(\lambda_k - \mu_k) + (1-\delta)\mu_k]}{1-\delta}|z|^{k-1} \leq \frac{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]}{B-A}$$

$$|z| = \left[\frac{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)](1-\delta)}{(B-A)[(\lambda_k-\mu_k)+\mu_k(1-\delta)]}\right]^{\frac{1}{k-1}}$$

By choosing $\Phi(z) = \dfrac{z}{(1-z)^2}$ and $\Psi(z) = z$, in the above theorem we get the following corollary.

Corollary 2.8. *Let the function f defined by (1.2) be in the class* $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$.

Then f is close-to-convex of order $\delta\,(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_1$, where

$$r_1 = \inf_k \left[\frac{(1-\delta)c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]}{(B-A)k}\right]^{\frac{1}{k-1}}, \quad k \geq 2. \tag{2.19}$$

The result is sharp with the extremal function f given by (2.3).

By choosing $\Phi(z) = \dfrac{z}{(1-z)^2}$ and $\Psi(z) = \dfrac{z}{1-z}$, in the above theorem we get the following corollary.

Corollary 2.9. *Let the function f defined by (1.2) be in the class $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$. Then f is starlike of order $\delta\,(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_2$, where*

$$r_2 = \inf_k \left[\frac{(1-\delta)c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]}{(B-A)(k-\delta)}\right]^{\frac{1}{k-1}}, \quad k \geq 2. \tag{2.20}$$

The result is sharp with the extremal function f given by (2.3).

By choosing $\Phi(z) = \dfrac{z+z^2}{(1-z)^3}$ and $\Psi(z) = \dfrac{z}{(1-z)^2}$, in the above theorem we get the following corollary.

Corollary 2.10. *Let the function f be defined by (1.2) be in the class $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$. Then f is convex of order $\delta\,(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_3$, where*

$$r_3 \leq \inf_k \left[\frac{(1-\delta)c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]}{k(B-A)(k-\delta)}\right]^{\frac{1}{k-1}}, \quad k \geq 2. \tag{2.21}$$

and $c^{\zeta}_k(m,\lambda)$ is given by (2.2), $k \geq 2$.

3. Convolution Properties

Theorem 3.1. *If f is of the form (1.2) and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ belongs to the class $\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$ then $h(z) = (f \bullet g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k$ will be an element of*

$\mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$ *with* $-1 \le A_1 < B_1 \le 1$, *where* $B_1 \ge \dfrac{A_1+\alpha}{1-\alpha}$, $A_1 \le 1-2\alpha$, *where*

$$\alpha = \frac{(B-A)^2}{[2B-A+1]^2(1+c_j^m(\lambda))^{\zeta}(1+\gamma c_j^m(\lambda)) - (B-A)^2}, \quad 0 \le \gamma \le 1$$

and these bounds are sharp.

Proof. From (2.1), we have

$$\sum_{k=2}^{\infty} c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]a_k \le B-A.$$

where $c_k(\zeta,\lambda) = [1+c_j^m(\lambda)(k-1)]^m$.

$$\sum_{k=2}^{\infty} \frac{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]a_k}{B-A} \le 1. \tag{3.1}$$

and

$$\sum_{k=2}^{\infty} \frac{c_k(\zeta,\lambda)[k(1+B)-(1+A)][1+\gamma c_j^m(\lambda)(k-1)]b_k}{B-A} \le 1. \tag{3.2}$$

In view of Cauchy-Schwarz inequality, from (3.1) and (3.2) we obtain

$$\sum_{k=2}^{\infty} u c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]\sqrt{a_k b_k} \le 1 \quad \text{where} \quad u = \frac{[k(1+B)-(1+A)]}{B-A} \tag{3.3}$$

We need to find A_1 and B_1 such that $h = f \bullet g \in \mathcal{T}^{\zeta}_{m,\lambda}(A,B,\gamma)$, or equivalently,

$$\sum_{k=2}^{\infty} u_1 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]a_k b_k \le 1 \text{ where } u_1 = \frac{[k(1+B_1)-(1+A_1)]}{B_1-A_1} \tag{3.4}$$

The inequality (3.4) will be true if
$u_1 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]a_k b_k \le u c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]\sqrt{a_k b_k}$ or

$$\sqrt{a_k b_k} \le \frac{u}{u_1}. \tag{3.5}$$

But from (3.3) we get $\sqrt{a_k b_k} \le \dfrac{1}{u c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]}$.

Thus (3.5) will be true if

$$\frac{1}{u c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]} \le \frac{u}{u_1},$$

or

$$u_1 \leq u^2 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)],$$

that is,

$$\frac{[k(1+B_1)-(1+A_1)]}{(B_1-A_1)} \leq u^2 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]. \tag{3.6}$$

Using $-1 \leq B < A \leq 1$, it is obvious that
$u^2 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)] > 1$, for $\zeta \geq 2$ and hence (3.6) yields

$$\frac{(B_1-A_1)}{1+B_1} \geq \frac{k-1}{u^2 c_k(\zeta,\lambda)[1+\gamma c_j^m(\lambda)(k-1)]-1} = \phi(k).$$

Since $\phi(k)$ is a decreasing function of k, for $k \geq 2$, $\phi(2)$ is the maximum value of $\phi(k)$ and therefore

$$\frac{B_1-A_1}{1+B_1} \geq \frac{(B-A)^2}{[2B-A+1]^2(1+c_j^m(\lambda))^\zeta(1+\gamma c_j^m(\lambda))-(B-A)^2} = \alpha. \tag{3.7}$$

Obviously, $\alpha < 1$ and fixing A_1 in (3.7), we get

$$B_1 \geq \frac{A_1+\alpha}{1-\alpha}. \tag{3.8}$$

Substituting $B_1 \leq 1$ in (3.8) we obtain, $A_1 \leq 1-2\alpha$.
For verifying sharpness, we note that if $k=2$, then

$$c_2(\zeta,\lambda) = (1+c_j^m(\lambda))^\zeta$$

and hence by taking

$$f(z) = g(z) = z - \frac{(B-A)(1+c_j^m(\lambda))^\zeta}{(2B-A+1)(1+\gamma c_j^m(\lambda))} z^2 \in \mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$$

we get

$$h(z) = (f \bullet g)(z) = z - \frac{(B-A)^2(1+c_j^m(\lambda))^\zeta}{(2B-A+1)^2(1+\gamma c_j^m(\lambda))^2} z^2 \in \mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma).$$

If we use this function h in (3.4), the inequality (3.7) transforms to equality, that is, $\dfrac{B_1-A_1}{1+B_1} = \alpha$. Now if $B_1 = 1$, then $A_1 = 1-\alpha$, which shows in this case $h \in \mathcal{T}_{m,\lambda}^{\zeta}(A,B,\gamma)$.

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